Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2006) 23–38
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

On the relationship of gerbes to the odd families index theorem

Alan L. Carey*, Bai-Ling Wang

Mathematical Sciences Institute, Australian National University, Canberra ACT 0200, Australia

Received 14 July 2004; received in revised form 7 October 2005; accepted 28 January 2006
Available online 10 March 2006

Abstract

The goal of this paper is to apply the universal gerbe of [A. Carey, J. Mickelsson, A gerbe obstruction to quantization of fermions on odd dimensional manifolds, Lett. Math. Phys. 51 (2000) 145–160] and [A.L. Carey, J. Mickelsson, The universal gerbe, Dixmier–Douady classes and gauge theory, Lett. Math. Phys. 59 (2002) 47–60] to give an alternative, simple and more unified view of the relationship between index theory and gerbes. We discuss determinant bundle gerbes [A. Carey, J. Mickelsson, M. Murray, Index theory, gerbes, and Hamiltonian quantization, Comm. Math. Phys. 183 (1997) 707–722] and the index gerbe of [J. Lott, Higher-degree analogs of the determinant line bundle, Comm. Math. Phys. 230 (1) (2002) 41–69] for the case of families of Dirac operators on odd dimensional closed manifolds. The method also works for a family of Dirac operators on odd dimensional manifolds with boundary, for a pair of Melrose and Piazza's $Cl(1)$-spectral sections for a family of Dirac operators on even dimensional closed manifolds with vanishing index in K-theory and, in a simple case, for manifolds with corners. The common feature of these bundle gerbes is that there exists a canonical bundle gerbe connection whose curving is given by the degree 2 part of the even eta form (up to a locally defined exact form) arising from the local family index theorem.

Keywords: Bundle gerbe; Index theory; K-theory; Dirac operators

1. Introduction

To the authors' knowledge the subject started with an unpublished manuscript of Segal [46] who described a substitute, for families of Dirac operators on odd dimensional manifolds, for Quillen's determinant line bundle. Subsequently in [18] Segal's construction was recognized as defining a bundle gerbe in the sense of Murray [38]. Murray's bundle gerbes are differential geometric objects which offer an alternative to Brylinski's description [14] of Giraud's gerbes. In [19] an explicit formula for the Dixmier–Douady class of the 'determinant bundle gerbe' of [18] was derived from the odd local index theorem for particular families of Dirac operators. We became aware of the existence of a more general point of view in discussions with Mathai on his work with Melrose [32]. This theory was realized in [29], where Lott gave a construction of the higher degree analogs of the determinant line bundle: the 'indexgerbe' and defined connective structures on them, for the case of a family of Dirac operators on odd dimensional closed spin manifolds. He also recognized the role of eta forms.

* Corresponding author.
E-mail addresses: acarey@maths.anu.edu.au (A.L. Carey), wangb@maths.anu.edu.au (B.-L. Wang).

0393-0440/$ - see front matter © 2006 Elsevier B.V. All rights reserved.
doi:10.1016/j.geomphys.2006.01.009

The so-called universal gerbe was introduced in [21,22] where Carey and Mickelsson analysed the obstruction to obtaining a second quantization for a smooth family of Dirac operators on an odd dimensional spin manifold with boundary. Explicit computations of the Dixmier–Douady class for the universal gerbe were also given. With hindsight [29] may be seen as an example of the universal gerbe for the case of families of Dirac operators. The index gerbe is studied from a slightly different point of view by Bunke in [15].

In this paper we use the universal gerbe to simplify the discussion of [29] and to provide a way to extend [29]. We also show that the universal gerbe can be used to construct other examples of geometrically interesting gerbes such as that for a family of Dirac operators on odd dimensional manifolds with boundary and for a pair of Melrose–Piazza $Cl(1)$-spectral sections for a family of Dirac operators on even dimensional closed manifolds with vanishing index in K-theory. These latter gerbes exhaust all bundle gerbes corresponding to the Dixmier–Douady classes in the image of the Chern character map on the K^1-group of the underlying manifold. In all these examples of gerbes, there exists a canonical bundle gerbe connection whose curving, up to an exact 2-form, is given by the degree 2 part of the even eta form arising from the local family index theorem. This enables us to propose an approach to family index theory manifolds with corners by using the bundle gerbe whose curving is given by the degree 2 part of the eta form.

For a single manifold with corners, Fredholm perturbations of Dirac-type operators and their index are thoroughly studied by Loya and Melrose in [31]. A more general and ambitious approach to these latter questions is contained in [16] which motivated us to develop the point of view of this paper.

As this paper is largely about gaining a unifying perspective we need to review earlier work. Section 2 summarizes what is needed from the theory of bundle gerbes. Section 3 describes briefly the family index theorems developed by Bismut [7], Bismut and Freed [10], Bismut and Cheeger [9], Mazzeo and Piazza [33] and Melrose and Piazza [34,35]. This section is required to set up notation. In Section 4, we recall the construction of the determinant bundle gerbe in [19] and study its geometry in Theorem 4.1. In Section 5, we give a complete proof of the existence theorem (Theorem 5.1) of a canonical bundle gerbe class over a base manifold associated with any element in K^1 of the base manifold using the universal gerbe. The results in this section make explicit some folklore of the subject.

In Section 6, we apply the constructions in Section 5 to the index gerbe associated with a family of Dirac operators on an odd dimensional manifold with or without boundary. This gives a simple construction of the gerbe discussed in [29,15]. Two new examples of bundle gerbes are obtained, one for a family of Dirac operators on odd dimensional manifolds with boundary, and the other for a pair of Melrose–Piazza $Cl(1)$-spectral sections for a family of Dirac operators on even dimensional closed manifolds with vanishing index in K-theory. Our main observation in this paper is that the universal gerbe suggests an approach to understanding the local family index theory (in particular the transgression of eta forms) for manifolds with codimension 2 corners. Our approach, if it can be fully developed, may provide an alternative to the treatment of index theory on manifolds with boundary and corners in [16,17,30].

2. Review of bundle gerbes

Our reference for this section is [38]. Take a smooth surjective submersion $\pi : Y \to M$ and let $Y^{[2]}$ be the fiber product associated with π with the obvious groupoid structure. A bundle gerbe $\mathcal{G}$ over M, as defined in [38], consists of $\pi : Y \to M$, and a principal $U(1)$-bundle $\mathcal{G}$ over $Y^{[2]} = Y \times_\pi Y$ together with a groupoid multiplication on $\mathcal{G}$, which is compatible with the natural groupoid multiplication on $Y^{[2]}$. The multiplication is represented by an isomorphism

$$m : \qquad \pi_1^*\mathcal{G} \otimes \pi_3^*\mathcal{G} \to \pi_2^*\mathcal{G} \tag{1}$$

of principal $U(1)$-bundles over $Y^{[3]} = Y \times_\pi Y \times_\pi Y$, where π_i, $i = 1, 2, 3$, are the three natural projections from $Y^{[3]}$ to $Y^{[2]}$ obtained by omitting the entry in position i for π_i.

In [39] it is shown that the natural equivalence relation on bundle gerbes is that of stable equivalence. Within each stable equivalence class we may choose a local representative by letting $\{U_\alpha\}$ be a good cover. Then the submersion π admits a local section s_α over U_α. The local bundle gerbe can be described by a family of local $U(1)$-bundles $\{\mathcal{G}_{\alpha\beta}\}$ over $\{U_{\alpha\beta} = U_\alpha \cap U_\beta\}$, the pull-back of $\mathcal{G}$ via (s_α, s_β). In this picture the bundle gerbe multiplication (1) is given by an isomorphism

$$\phi_{\alpha\beta\gamma} : \qquad \mathcal{G}_{\alpha\beta} \otimes \mathcal{G}_{\beta\gamma} \cong \mathcal{G}_{\alpha\gamma}, \tag{2}$$

over $U_{\alpha\beta\gamma}$, such that $\phi_{\alpha\beta\gamma}$ is associative as a groupoid multiplication. A related local picture of gerbes was introduced by Hitchen [27].

A Cěch cocycle $\{f_{\alpha\beta\gamma}\}$ can be obtained from the isomorphism (2) by choosing a section $s_{\alpha\beta}$ of $\mathcal{G}_{\alpha\beta}$, i.e.,

$$\phi_{\alpha\beta\gamma}(s_{\alpha\beta} \otimes s_{\beta\gamma}) = f_{\alpha\beta\gamma} \cdot s_{\alpha\gamma}$$

for a $U(1)$-valued function $f_{\alpha\beta\gamma}$ over $U_{\alpha\beta\gamma}$. The equivalence class of $\{f_{\alpha\beta\gamma}\}$ does not depend on the choice of local sections $\{s_{\alpha\beta}\}$, and represents the Dixmier–Douady class of the bundle gerbe in

$$H^2_{\text{Cěch}}(M, \underline{U(1)}) \cong H^3(M, \mathbb{Z}),$$

where $\underline{U(1)}$ is the sheaf of continuous $U(1)$-valued functions over M.

The geometry of bundle gerbes, bundle gerbe connections and their curvings is studied in [38]. On the corresponding local bundle gerbe a bundle gerbe connection is a family of $U(1)$-connections $\{\nabla_{\alpha\beta}\}$ on $\{\mathcal{G}_{\alpha\beta}\}$ which is compatible with the isomorphism (2), i.e., $A_{\alpha\beta} = \nabla_{\alpha\beta} s_{\alpha\beta}/s_{\alpha\beta}$ satisfies

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = f^{-1}_{\alpha\beta\gamma} \mathrm{d} f_{\alpha\beta\gamma}. \tag{3}$$

A curving then is a locally defined 2-form B_α such that,

$$B_\alpha - B_\beta = \mathrm{d}A_{\alpha\beta}, \tag{4}$$

over $U_\alpha \cap U_\beta$. We remark that in string theory applications this system $\{B_\alpha\}_\alpha$ of local 2-forms is called the B-field. We will use this terminology here. The B-field is unique, up to locally defined exact 2-forms. If we choose another representative for the B-field $\{B_\alpha + dC_\alpha\}_\alpha$, then we need to modify the gerbe connection by adding $\{C_\alpha - C_\beta\}$ to $\{A_{\alpha\beta}\}$. This has no effect on the compatibility condition (3). This local description of connection and curving for a local bundle gerbe defines an element

$$[(f_{\alpha\beta\gamma}, A_{\alpha\beta}, B_\alpha)]$$

in the degree 3 Deligne cohomology group $H^3_{\text{Del}}(M, \mathcal{D}^3)$. The curvature of B-field is given by $\{\mathrm{d}B_\alpha\}$ and we note that

$$\mathrm{d}B_\alpha = \mathrm{d}B_\beta, \qquad \text{on } U_{\alpha\beta}$$

is a globally defined 3-form which represents the image of the Dixmier–Douady class in $H^3(M, \mathbb{R})$ up to a factor of $\frac{\mathrm{i}}{2\pi}$. This 3-form is the gerbe analogue of the Chern class of line bundles with connections. In this paper, we often suppress the normalization factor and identify the bundle gerbe curvature with the differential form representing the image of the Dixmier–Douady class in $H^3(M, \mathbb{R})$.

In fact, though we do not use it here, the B-field of a bundle gerbe $\mathcal{G}$ over M is a Hopkins–Singer differential cocycle as defined in [28] whose equivalence class is the corresponding degree 3 Deligne cohomology class. One has in fact the following commutative diagram

$$\begin{array}{ccc} H^3_{\text{Del}}(M, \mathcal{D}^3) & \longrightarrow & H^3(M, \mathbb{Z}) \\ \downarrow & & \downarrow \\ \Omega^3_{\mathbb{Z}}(M) & \longrightarrow & H^3(M, \mathbb{R}) \end{array}$$

where $\Omega^3_{\mathbb{Z}}(M)$ is the space of closed 3-forms on M with integral periods. We also note that there have been other applications of bundle gerbes to various aspects of string theory and quantum field theory: [13,12,18–22,24,25,36].

Important remark. There is a subtlety here relating to the gauge transformations of the B-field. To illustrate this subtlety, given a globally defined closed 2-form ω (not necessarily of integer periods), with respect to a cover $\{U_\alpha\}$ of M we can write $\omega|_{U_\alpha} = \mathrm{d}\theta_\alpha$ as an exact 2-form by the Poincaré Lemma. Then on the one hand, one can see that

$$(1, 0, \omega|_{U_\alpha})$$

represents a trivial degree 3 Deligne class in $H^3_{\mathrm{Del}}(M, \mathcal{D}^3)$ if and only if the closed 2-form ω has integer periods. This claim follows from the exact sequence

$$0 \longrightarrow \Omega^2(M)/\Omega^2_{\mathbb{Z}}(M) \longrightarrow H^3_{\mathrm{Del}}(M, \mathcal{D}^3) \longrightarrow H^3(M, \mathbb{Z}) \longrightarrow 0.$$

In this sense, the gauge transformation of the B-field is given by line bundles with connection on M. On the other hand,

$$(1, \theta_\alpha - \theta_\beta, \omega|_{U_\alpha}) = (1, \theta_\alpha - \theta_\beta, \mathrm{d}\theta_\alpha)$$

always represents a trivial degree 3 Deligne class in $H^3_{\mathrm{Del}}(M, \mathbb{Z})$, as $(1, \theta_\alpha - \theta_\beta, \mathrm{d}\theta_\alpha)$ is a coboundary element. In this paper, a curving on a bundle gerbe is always defined up to a locally defined exact 2-form in this latter sense so that the corresponding B-field is well defined up to a degree 3 Deligne coboundary term of form $(1, \theta_\alpha - \theta_\beta, \mathrm{d}\theta_\alpha)$. Here $\mathrm{d}\theta_\alpha$ may not be a globally defined exact 2-form. We hope that this remark will clarify any confusion when we talk about curvings on a bundle gerbe, such as the degree 2 part of the even eta form up to an exact 2-form.

3. Review of local family index theory

In this section, we briefly review local index theory and eta forms and set up notation. Local family index theorems refine the cohomological family index theorem of Atiyah and Singer, while eta forms transgress the local family index theorem at the level of differential forms. The main reference for this section is Bismut's survey paper [6] and the references therein [1,4,5,8,11,40,41].

Let $\pi : X \to B$ be a smooth fibration over a closed smooth manifold B, whose fibers are diffeomorphic to a compact, oriented, spin manifold M. Let (V, h^V, ∇^V) be a Hermitian vector bundle over X equipped with a unitary connection.

Let $g^{X/B}$ be a metric on the relative tangent bundle $T(X/B)$ (the vertical tangent subbundle of TX), which is fiberwise a product metric near the boundary if the boundary of M is non-empty. In the latter case, $\partial X \to B$ is also a fibration with compact fiber ∂M. Let $S_{X/B}$ be the corresponding spinor bundle associated with the spin structure on $(T(X/B), g^{X/B})$.

Let $T^H X$ be a smooth vector subbundle of TX (a horizontal vector subbundle of TX), such that

$$TX = T^H X \oplus T(X/B), \tag{5}$$

and such that if ∂M is non-empty, $T^H X|_{\partial X} \subset T(\partial X)$. Then $T^H(\partial X) = T^H X|_{\partial X}$ is a horizontal subbundle of $T(\partial X)$.

Denote by P^v the projection of TX onto the vertical tangent bundle under the decomposition (5). Theorem 1.9 of [7] shows that $(T^H X, g^{X/B})$ determines a canonical Euclidean connection $\nabla^{X/B}$ as follows. Choose a metric g^X on TX such that $T^H X$ is orthogonal to $T(X/B)$ and such that $g^{X/B}$ is the restriction of g^X to $T(X/B)$. Then

$$\nabla^{X/B} = P^v \circ \nabla^X$$

where ∇^X is the Levi-Civita connection on (TX, g^X).

Write

$$\hat{A}(x) = \frac{x/2}{\sinh(x/2)}, \qquad \mathrm{Ch}(x) = \exp(x).$$

Then the characteristic classes can be represented by closed differential forms on X as

$$\hat{A}(T(X/B), \nabla^{X/B}) = \det{}^{1/2}\left(\hat{A}\left(\frac{\mathrm{i}(\nabla^{X/B})^2}{2\pi}\right)\right),$$

$$\mathrm{Ch}(V, \nabla^V) = \mathrm{Tr}\left(\mathrm{Ch}\left(\frac{\mathrm{i}(\nabla^V)^2}{2\pi}\right)\right).$$

For any $b \in B$, let $\not{D}_b$ be the Dirac operator acting on $C^\infty(X_b, (S_{X/B} \otimes V)|_{X_b})$, where $X_b = \pi^{-1}(b)$ is the fiber of π over b. Then $\{\not{D}_b\}_{b\in B}$ is a family of elliptic operators on the fibers of π, acting fiberwise on an infinite dimensional vector space $\mathbb{C}^\infty(X_b, (S_{X/B} \otimes V)|_{X_b})$, parametrized by $b \in B$. With respect to the inner product from the Hermitian metrics on the spinor bundle $S_{X/B}$ and V, $\{\not{D}_b\}_{b\in B}$ can be viewed as a family of unbounded operators

$$\not{D}_b : L^2(X_b, (S_{X/B} \otimes V)|_{X_b}) \longrightarrow L^2(X_b, (S_{X/B} \otimes V)|_{X_b}).$$

Remark 3.1. (a) If the fiber of $\pi : X \to B$ is even dimensional, then the spinor bundle $S_{X/B} = S^+_{X/B} \oplus S^-_{X/B}$ is $\mathbb{Z}_2$-graded, and

$$\not{D}_b = \begin{pmatrix} 0 & \not{D}^*_{+,b} \\ \not{D}_{+,b} & 0 \end{pmatrix}.$$

Then $\not{D}_{+,b}(1+\not{D}^*_{+,b}\not{D}_{+,b})^{-\frac{1}{2}}$ is a continuous family of Fredholm operators, which defines a K^0-element, denoted by $\mathrm{Ind}(\not{D}_+)$ in $K^0(B)$. The Atiyah–Singer index theorem becomes

$$\mathrm{Ch}(\mathrm{Ind}(\not{D}_+)) = \pi_*[\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)] \in H^{\mathrm{even}}(B, \mathbb{R}) \tag{6}$$

where π_* is given by the integration along the fibers on the level of differential forms.

(b) If the fiber of $\pi : X \to B$ is odd dimensional, the family $\{\not{D}_b\}_{b\in B}$ defines a K^1-element $\mathrm{Ind}(\not{D}) \in K^1(B)$. The Atiyah–Singer index theorem becomes

$$\mathrm{Ch}(\mathrm{Ind}(\not{D})) = \pi_*[\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)] \in H^{\mathrm{odd}}(B, \mathbb{R}). \tag{7}$$

Now assume that the fibers of the fibration $\pi : X \to B$ are diffeomorphic to an even dimensional manifold M with non-empty boundary ∂M, then the family of boundary Dirac operators $\{\not{D}^{\partial M}_b\}_{b\in B}$ for the boundary fibration $\partial\pi : \partial X \to B$ defines a zero class in $K^1(B)$ by cobordism invariance of the Atiyah–Singer index theory. Hence, there exists an even eta form $\tilde{\eta}_{\mathrm{even}}$ on B satisfying the transgression formula

$$d\tilde{\eta}_{\mathrm{even}} = (\partial\pi)_*(\hat{A}(T(\partial X/B), \nabla^{\partial X/B})\mathrm{Ch}(V, \nabla^V)),$$

$\tilde{\eta}_{\mathrm{even}}$ is unique up to an exact odd form on B.

Melrose and Piazza introduced the notion of a spectral section for the family of boundary Dirac operators $\{\not{D}^{\partial M}_b\}_{b\in B}$ in [34], where a spectral section is a smooth family of self-adjoint pseudo-differential projections

$$P_b : \qquad L^2(\partial X_b, (S_{\partial X/B} \otimes V)|_{\partial X_b}) \to L^2(\partial X_b, (S_{\partial X/B} \otimes V)|_{\partial X_b}),$$

such that there is $R > 0$ for which $\not{D}^{\partial M}_b s = \lambda s$ implies that

$$\begin{aligned} P_b s &= s \qquad \lambda > R, \\ P_b s &= 0 \qquad \lambda < -R. \end{aligned}$$

For a fixed spectral section P, then Melrose and Piazza constructed an even eta form $\tilde{\eta}^P_{\mathrm{even}}$ depending only on $\{\not{D}^{\partial M}_b\}_{b\in B}$ and the spectral section P. With the boundary condition given by a spectral section P, they also established the local family index theorem for the family of Dirac operators $\{\not{D}^M_{+,P;b}\}_{b\in B}$ in [34].

Theorem 3.2. *Let $\pi : X \to B$ be a smooth fibration with fiber diffeomorphic to an even dimensional spin manifold M with boundary. The family of Dirac operators $\{\not{D}^M_{+,P;b}\}_{b\in B}$ has a well-defined index $\mathrm{Ind}(\not{D}^M_{+,P}) \in K^0(B)$, with the Chern character given by*

$$\mathrm{Ch}(\mathrm{Ind}(\not{D}^M_{+,P})) = \int_M \hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V) - \tilde{\eta}^P_{\mathrm{even}} \in H^{\mathrm{even}}(B, \mathbb{R}).$$

When the fibers of the fibration $\pi : X \to B$ are diffeomorphic to an odd dimensional manifold M with non-empty boundary, the family of boundary Dirac operator $\{\not{D}^{\partial M}_b\}_{b\in B}$ consists of self-adjoint, elliptic, odd differential operators acting on

$$L^2(X_b, (S^+_{\partial X/B} \otimes V)|_{\partial X_b}) \oplus L^2(X_b, (S^-_{\partial X/B} \otimes V)|_{X_b}). \tag{8}$$

In order to get a continuous family of self-adjoint Fredholm operators, Melrose and Piazza introduced a $Cl(1)$-spectral section P and constructed an odd eta form $\tilde{\eta}^P_{\mathrm{odd}}$ which, modulo exact forms, depends only on $\{\not{D}^{\partial M}_b\}_{b\in B}$ and the $Cl(1)$-spectral section [35]. The $Cl(1)$-condition (where $Cl(1)$ denotes the Clifford algebra $Cl(\mathbb{R})$) is given by

$$c(\mathrm{d}u) \circ P + P \circ c(\mathrm{d}u) = c(\mathrm{d}u),$$

with u being the inward normal coordinate near the boundary ∂M.

Theorem 3.3 ([35]). *Let $\pi : X \to B$ be a smooth fibration with fiber diffeomorphic to an odd dimensional spin manifold M with boundary. With the boundary condition given by a $Cl(1)$-spectral section P, the family of Dirac operators $\{\not{D}^M_{P;b}\}_{b\in B}$ has a well-defined index*

$$\mathrm{Ind}(\not{D}^M_P) \in K^1(B),$$

with the Chern character form given by

$$\mathrm{Ch}(\mathrm{Ind}(\not{D}^M_P)) = \int_M \hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V) - \tilde{\eta}^P_{\mathrm{odd}} \in H^{\mathrm{odd}}(B, \mathbb{R}). \tag{9}$$

4. Determinant bundle gerbe

In [19], Carey, Mickelson and Murray used the determinant bundle gerbe to study the bundle of fermionic Fock spaces parametrized by connections on principal G-bundles on odd dimensional manifolds (for G a compact Lie group). We briefly review this construction here using the local family index theorem.

Let M be a closed, oriented, spin manifold with a Riemannian metric g^M. Let S be the spinor bundle over M and let (V, h^V) be a Hermitian vector bundle. Denote by $\mathcal{A}$ the space of unitary connections on V and by $\mathcal{G}$ the based gauge transformation group, that is, those gauge transformations fixing the fiber over some fixed point in M. With proper regularity on connections, the quotient space $\mathcal{A}/\mathcal{G}$ is a smooth Frechet manifold.

Let $\lambda \in \mathbb{R}$, and let

$$U_\lambda = \{A \in \mathcal{A} | \lambda \notin \mathrm{spec}(\not{D}_A)\},$$

where the Dirac operator $\not{D}_A$ acts on $C^\infty(M, S \otimes V)$. Let H be the space of square integrable sections of $S \otimes V$. For any $A \in U_\lambda$, there is a uniform polarization:

$$H = H_+(A, \lambda) \oplus H_-(A, \lambda)$$

given by the spectral decomposition of H with respect to $\not{D}_A - \lambda$ into the eigenspaces corresponding to positive and negative spectra. Denote by $P_{H_\pm(A,\lambda)}$ the orthogonal projection onto $H_\pm(A, \lambda)$.

Fix $\lambda_0 \in \mathbb{R}$ and a reference connection $A_0 \in \mathcal{A}$ such that the Dirac operator $\not{D}_{A_0} - \lambda_0$ is invertible and denote by $P_{H_\pm(A_0,\lambda_0)}$ the orthogonal projection onto $H_\pm(A_0, \lambda_0)$. Over U_λ, there exists a complex line bundle Det_λ, which is essentially the determinant line bundle for the even dimensional Dirac operators on $[0, 1] \times M$ coupled to some path of connections in $\mathcal{A}$ with respect to the generalized Atiyah–Singer–Patodi boundary condition [2]. The path of connections $A(t)$ can chosen to be any smooth path connecting A_0 and $A \in U_\lambda$, for simplicity, we take

$$\mathbb{A} = A(t) = f(t)A + (1 - f(t))A_0 \tag{10}$$

for $t \in [0, 1]$ where f is zero on $[0, 1/4]$, equal to one on $[3/4, 1]$ and interpolates smoothly between these values on $[1/4, 3/4]$. The generalized Atiyah–Singer–Patodi boundary condition is determined by the orthogonal spectrum projection

$$P_\lambda = P_{H_+(A_0,\lambda_0)} \oplus P_{H_-(A,\lambda)},$$

that is, the even dimensional Dirac operator on $[0, 1] \times M$ acts on the spinor fields whose boundary components belong to $H_-(A_0, \lambda_0)$ at $\{0\} \times M$ and $H_+(A, \lambda)$ at $\{1\} \times M$.

Note that P_λ can be thought as a spectral section in the sense of Melrose and Piazza [34] for the family of Dirac operators

$$\{\not{D}_{A_0} \oplus \not{D}_A\}_{A\in U_\lambda},$$

on $\{0\} \times M \sqcup \{1\} \times M$. Then the eta form $\tilde{\eta}^{P_\lambda}_{\mathrm{even}}$ associated with $\{\not{D}_{A_0} \oplus \not{D}_A\}_{A\in U_\lambda}$ and the spectral section P_λ is an even differential form on U_λ, which is unique up to an exact form.

Denote by $\not{D}^{P_\lambda}_{\mathbb{A}}$ the even dimensional Dirac operator $\not{D}_{\mathbb{A}}$ with respect to the APS-type boundary condition P_λ. Then the even dimensional Dirac operator

$$\not{D}^{P_\lambda}_{\mathbb{A}} = \begin{pmatrix} 0 & \not{D}^{P_\lambda}_{\mathbb{A},-} \\ \not{D}^{P_\lambda}_{\mathbb{A},+} & 0 \end{pmatrix}$$

is a Fredholm operator, whose determinant line is given by

$$\mathrm{Det}_\lambda(A) = \Lambda^{\mathrm{top}}(\mathrm{Ker}\,\not{D}^{P_\lambda}_{\mathbb{A},+})^* \otimes \Lambda^{\mathrm{top}}(\mathrm{Coker}\,\not{D}^{P_\lambda}_{\mathbb{A},+}).$$

The family of Fredholm operators $\{\not{D}^{P_\lambda}_{\mathbb{A}}\}$, parametrized by $A \in U_\lambda$, defines a determinant line bundle over U_λ, which is given by

$$\mathrm{Det}_\lambda = \bigcup_{A\in U_\lambda} \mathrm{Det}_\lambda(A).$$

The determinant line bundle Det_λ can be equipped with a Quillen metric and a Bismut–Freed unitary connection whose curvature can be calculated by the local family index theorem.

Over $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$, there exists a complex line bundle $\mathrm{Det}_{\lambda\lambda'}$ such that

$$\mathrm{Det}_{\lambda\lambda'} = \mathrm{Det}^*_\lambda \otimes \mathrm{Det}_{\lambda'}.$$

These local line bundles $\{\mathrm{Det}_{\lambda\lambda'}\}$ over $\mathcal{A}$ form a bundle gerbe as established in [19]. As $\mathcal{A}$ is contractible this bundle gerbe is trivial so the interest in [19] lies in what happens when one takes the induced bundle gerbe on $\mathcal{A}/\mathcal{G}$.

In the following, we will apply the local family index theorem to study the geometry of this determinant bundle gerbe. To apply the local family index theorem, we should restrict ourselves to a smooth finite dimensional submanifold of $\mathcal{A}$. For convenience however, we will formally work on the infinite dimensional manifold $\mathcal{A}$ directly.

Consider the trivial fibration $[0, 1] \times M \times \mathcal{A}$ over $\mathcal{A}$ with fiber $[0, 1] \times M$ an even dimensional manifold with boundary. Over $[0, 1] \times M \times \mathcal{A}$, there is a Hermitian vector bundle $\mathbb{V}$ which is the pull-back bundle of V.

There is a universal unitary connection on $\mathbb{V}$, also denoted by $\mathbb{A}$, whose vector potential at (t, x, A) is given by

$$\mathbb{A}(t, x) = A(t)(x),$$

where $A(t)$ is given by (10). Denote by $\mathrm{Ch}(\mathbb{V}, \mathbb{A})$ the Chern character of $(\mathbb{V}, \mathbb{A})$.

Now we can state the following theorem regarding the geometry of the bundle gerbe over $\mathcal{A}/\mathcal{G}$ constructed in [19].

Theorem 4.1. *The local line bundles $\{\mathrm{Det}_{\lambda\lambda'}\}$ descend to local line bundles over $\mathcal{A}/\mathcal{G}$, which in turn define a local bundle gerbe over $\mathcal{A}/\mathcal{G}$. Moreover, the induced unitary connection and the even eta form $\tilde{\eta}^{P_\lambda}_{\mathrm{even}}$ (up to an exact 2-form) descend to a bundle gerbe connection and curving on the local bundle gerbe over $\mathcal{A}/\mathcal{G}$, whose bundle gerbe curvature is given by the differential form*

$$\left(\int_M \hat{A}(TM, \nabla^{TM})\mathrm{Ch}(\mathbb{V}, \mathbb{A})\right)_{(3)}.$$

Proof. Equip Det_λ with the Quillen metric and its unitary connection, then its first Chern class is represented by the degree 2 part of the following differential form:

$$\int_{[0,1]\times M} \hat{A}(TM, \nabla^{TM})\mathrm{Ch}(\mathbb{V}, \mathbb{A}) - \tilde{\eta}^{P_\lambda}_{\mathrm{even}}.$$

In this formula ∇^{TM} is the Levi-Civita connection on (TM, g^M), $\hat{A}(TM, \nabla^{TM})$ represents the $\hat{A}$-genus of M, and $\tilde{\eta}^{P_\lambda}_{\mathrm{even}}$ is the even eta form on U_λ associated with the family of boundary Dirac operators and the spectral section P_λ. Note that $\tilde{\eta}^{P_\lambda}_{\mathrm{even}}$ (modulo exact forms) is uniquely determined, see Theorem 3.2.

The induced connection on $\mathrm{Det}_{\lambda\lambda'}$ implies that its first Chern class is given by

$$(\tilde{\eta}^{P_\lambda}_{\mathrm{even}} - \tilde{\eta}^{P_{\lambda'}}_{\mathrm{even}})_{(2)}.$$

Note that the eta form $\tilde{\eta}^{P_\lambda}_{\text{even}}$ is unique up to an exact form (in an analogous way to the B-field), hence, $(\tilde{\eta}^{P_\lambda}_{\text{even}} - \tilde{\eta}^{P_{\lambda'}}_{\text{even}})_{(2)}$ is a well-defined element in $H^2(M, \mathbb{R})$. From the transgression formula for the eta forms and Stokes formula, we know that over $U_{\lambda\lambda'}$,

$$\mathrm{d}(\tilde{\eta}^{P_\lambda}_{\text{even}})_{(2)} - \mathrm{d}(\tilde{\eta}^{P_{\lambda'}}_{\text{even}})_{(2)} = \left(\int_M \hat{A}(TM, \nabla^{TM})\mathrm{Ch}(\mathbb{V}, \mathbb{A})\right)_{(3)}.$$

Here we use the fact that the contribution from $\{0\} \times M$ vanishes as a differential form on $\mathcal{A}$, as we fixed a connection A_0 over $\{0\} \times M$. We also use the same notation $(\mathbb{V}, \mathbb{A})$ to denote the Hermitian vector bundle over $M \times \mathcal{A}$ and the universal unitary connection $\mathbb{A}$.

As the gauge group acts on $\mathcal{A}$ and Det_λ covariantly, by quotienting out $\mathcal{G}$, we obtain the bundle gerbe over $\mathcal{A}/\mathcal{G}$ described in the theorem. □

Remark 4.2. The Dixmier–Douady class in Theorem 4.1 is often non-trivial. For example note that when $\dim M < 4$, there is no contribution from the $\hat{A}$-genus and in particular, for $\dim M = 1$ or 3, using the Chern–Simons forms, non-triviality was proved by explicit calculation in [19] (see also [23,37,41]).

The above construction can be generalized to the fibration case as in the local family index theorem for odd dimensional manifolds with boundary of Section 3. When the fibers are closed odd dimensional spin manifolds, this leads to the index gerbe as discussed by Lott [29]. In the next section, we discuss the universal bundle gerbe as in [21, 22], which provides a unifying viewpoint for those bundle gerbes constructed from various determinant line bundles.

5. The universal bundle gerbe

Let H be an infinite dimensional separable complex Hilbert space. Let $\mathcal{F}_*^{a.s}$ be the space of all self-adjoint Fredholm operators on H with positive and negative essential spectra. With the norm topology on $\mathcal{F}_*^{a.s}$, Atiyah and Singer [3] showed that $\mathcal{F}_*^{a.s}$ is a representing space for the K^1-group, that is, for any closed manifold B,

$$K^1(B) \cong [B, \mathcal{F}_*^{a.s}]$$

the homotopy classes of continuous maps from B to $\mathcal{F}_*^{a.s}$. As $\mathcal{F}_*^{a.s}$ is homotopy equivalent to $\mathcal{U}^{(1)}$, the group of unitary automorphisms $g : H \to H$ such that $g - 1$ is trace class, we obtain

$$K^1(B) \cong [B, \mathcal{U}^{(1)}].$$

In [21], a universal bundle gerbe was constructed on $\mathcal{U}^{(1)}$ with the Dixmier–Douady class given by the basic 3-form

$$\frac{1}{24\pi^2}\mathrm{Tr}(g^{-1}\mathrm{d}g)^3, \tag{11}$$

the generator of $H^3(\mathcal{U}^{(1)}, \mathbb{Z})$.

For any compact Lie group, this basic 3-form gives the so-called basic gerbe. We prefer however to call it the universal bundle gerbe in the sense that many examples of bundle gerbes on smooth manifolds are obtained by pulling back this universal bundle gerbe via certain smooth maps from B to $\mathcal{U}^{(1)}$.

Note that the odd Chern character of $K^1(B) = [B, \mathcal{U}^{(1)}]$ is given by

$$\mathrm{Ch}([g]) = \sum_{n\geq 0}\left[(-1)^n \frac{n!}{(2\pi i)^{n+1}(2n+1)!}\mathrm{Tr}((g^{-1}\mathrm{d}g)^{2n+1})\right], \tag{12}$$

for a smooth map $g : B \to \mathcal{U}^{(1)}$ representing a K^1-element $[g]$ in $K^1(B)$. This odd Chern character formula was proved in [26].

The following theorem was established in [21] as the first obstruction to obtaining a second quantization for a smooth family of Dirac operators (parametrized by B) on an odd dimensional spin manifold. By a second quantization, we mean an irreducible representation of the canonical anticommutative relations (CAR) algebra, the complex Clifford algebra $Cl(H \oplus \bar{H})$, which is compatible with the action of the quantized Dirac operator. Such a representation is given by the Fock space associated with a polarization on H. For a bundle of Hilbert spaces over B, a continuous polarization always exists locally, but not necessarily globally. This leads to a bundle gerbe over B.

Theorem 5.1. *For any K^1-element $[g] \in K^1(B)$ represented by a smooth map $g : B \to \mathcal{U}^{(1)}$. There exists a canonical construction of a bundle gerbe over B with connection and curving whose bundle gerbe curvature is given by the degree 3 part of*

$$\frac{1}{24\pi^2}\mathrm{Tr}(g^{-1}\mathrm{d}g)^3.$$

Proof. The bundle gerbe we are after is essentially the pull-back bundle gerbe from the universal bundle gerbe over $\mathcal{U}^{(1)}$ under the smooth map $g : B \to \mathcal{U}^{(1)}$.

Consider the polarized Hilbert space $\mathcal{H} = L^2(S^1, H)$ with the polarization ϵ given by the Hardy decomposition.

$$\mathcal{H} = L^2(S^1, H) = \mathcal{H}_+ \oplus \mathcal{H}_-.$$

That is we take the polarization given by splitting into positive and negative Fourier modes. Then the smooth based loop group $\Omega\mathcal{U}^{(1)}$ acts naturally on $\mathcal{H}$. From [42], we see that

$$\Omega\mathcal{U}^{(1)} \subset \mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon). \tag{13}$$

Here $\mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon)$ is the restricted unitary group of H with respect to the polarization ϵ, those $g \in U(\mathcal{H})$ such that the off-diagonal block of g is of Hilbert–Schmidt type. It was shown in [21] that the inclusion $\Omega\mathcal{U}^{(1)} \subset \mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon)$ is a homotopy equivalence.

We know that the holonomy map from the space of connections on a trivial $\mathcal{U}^{(1)}$-bundle over S^1 provides a model for the universal $\Omega\mathcal{U}^{(1)}$-bundle. Hence, $\mathcal{U}^{(1)}$ is a classifying space for $\Omega\mathcal{U}^{(1)}$.

From $K^1(B) \cong [B, \mathcal{U}^{(1)}]$, we conclude that elements in $K^1(B)$ are in one-to-one correspondence with isomorphism classes of principal $\mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon)$-bundles over B.

Associated with the basic three form (11) on $\mathcal{U}^{(1)}$ is the universal gerbe realized as the lifting bundle gerbe [38] associated with the central extension

$$1 \to U(1) \to \hat{\mathcal{U}}_{\mathrm{res}} \to \mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon) \to 1. \tag{14}$$

We recall some of the theory behind this fact. First note that the fundamental representation of $\hat{\mathcal{U}}_{\mathrm{res}}$ acts on the Fock space (see [42])

$$\mathcal{F}_{\mathcal{H}} = \Lambda(\mathcal{H}_+) \otimes \Lambda(\bar{\mathcal{H}}_-),$$

associated with the polarized Hilbert space $\mathcal{H} = \mathcal{H}_{\oplus}\mathcal{H}_-$. This gives rise to a homomorphism $\mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon) \to \mathbb{P}U(\mathcal{F}_{\mathcal{H}})$ which induces a $\mathbb{P}U(\mathcal{F}_{\mathcal{H}})$ principal bundle from the $\mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon)$ principal bundle associated with the K^1-element $[g]$. Denote by $\mathcal{P}_g$ the resulting $\mathbb{P}U(\mathcal{F}_{\mathcal{H}})$ principal bundle corresponding to a smooth map $g : B \to \mathcal{U}^{(1)}$ representing $[g]$. The corresponding lifting bundle gerbe (as defined in [38]) is the canonical bundle gerbe over B for which we are looking.

Recall that the lifting bundle gerbe associated with $\mathcal{P}_g$ can be described locally as follows [38]. Take a local trivialization of $\mathcal{P}_g$ with respect to a good cover of $B = \bigcup_\alpha U_\alpha$. Assume that over U_α, the trivialization is given by a local section

$$s_\alpha : \qquad U_\alpha \to \mathcal{P}_g|_{U_\alpha},$$

such that the transition function over $U_{\alpha\beta} = U_\alpha \cap U_\beta$ is given by a smooth function

$$\gamma_{\alpha\beta} : \qquad U_{\alpha\beta} \to \mathcal{U}_{\mathrm{res}}(\mathcal{H}, \epsilon) \subset \mathbb{P}U(\mathcal{F}_{\mathcal{H}}).$$

We can define a local Hermitian line bundle $\mathcal{L}_{\alpha\beta}$ over $U_{\alpha\beta}$ as follows. First pull back the principal $U(1)$ bundle (14) using $\gamma_{\alpha\beta}$. Then construct the associated line bundles $\{\mathcal{L}_{\alpha\beta}\}$ over the double intersections $U_{\alpha\beta}$. This family of local Hermitian line bundles $\{\mathcal{L}_{\alpha\beta}\}$ defines a bundle gerbe over B with multiplication obtained from the multiplication in $\hat{\mathcal{U}}_{\mathrm{res}}$.

In [21] a Hermitian connection on $\{\mathcal{L}_{\alpha\beta}\}$ is given which defines a bundle gerbe connection, together with a choice of a curving such that the bundle gerbe curvature is given by

$$\frac{1}{24\pi^2}\mathrm{Tr}(g^{-1}\mathrm{d}g)^3. \qquad \square$$

For a Hermitian vector bundle (V, h^V) over an oriented, closed, spin manifold M with a Riemannian metric denote by $\mathcal{A}$ the space of unitary connections on V and by $\mathcal{G}$ the based gauge transformation group as in Section 4.

In [22], an explicit smooth map $\tilde{g} : \mathcal{A}/\mathcal{G} \to \mathcal{U}^{(1)}$ was constructed: firstly assign to any $A \in \mathcal{A}$ a unitary operator in $\mathcal{U}^{(1)}$ by

$$A \to g(A) = -\exp\left(\mathrm{i}\pi \frac{\not{D}_A}{|\not{D}_A| + \chi(|\not{D}_A|)}\right),$$

where χ is any positive smooth exponentially decaying function on $[0, \infty)$ with $\chi(0) = 1$. As $g(A)$ is not gauge invariant,

$$g(A^u) = u^{-1}g(A)u,$$

for any $u \in \mathcal{G}$, it does not define a smooth map from $\mathcal{A}/\mathcal{G}$ to $\mathcal{U}^{(1)}$. In order to get a gauge invariant map, a global section for the associated $U(H)$-bundle $\mathcal{A} \times_{\mathcal{G}} U(H)$ is needed. This exists due to the contractibility of $U(H)$. This section is given by a smooth map $r : \mathcal{A} \to U(H)$ such that $r(A^u) = u^{-1}r(A)$. Then the required map $\tilde{g} : \mathcal{A}/\mathcal{G} \to \mathcal{U}^{(1)}$ is given by $\tilde{g}(A) = r(A)^{-1}g(A)r(A)$.

Let B be a smooth submanifold of $\mathcal{A}/\mathcal{G}$. The restriction of $\tilde{g}$ to B defines an element in $K^1(B)$, which is exactly the family of Dirac operators over B associated with the universal connection $\mathbb{A}$ on $\mathbb{V}$ (see Section 3 for the definition).

From the local family index theorem, we know that the curvatures satisfy

$$\left(\int_M \hat{A}(TM, \nabla^T M)\mathrm{Ch}(\mathbb{V}, \mathbb{A})\right)_{(3)} = \frac{1}{24\pi^2}\mathrm{Tr}(\tilde{g}^{-1}\mathrm{d}\tilde{g})^3. \tag{15}$$

Note that the left hand side was established in [19] and the right hand side in [22]. This relation fixes the image of the Dixmier–Douady class in $(H^3(B, \mathbb{R}))$. It follows that the bundle gerbe over $B \subset \mathcal{A}/\mathcal{G}$ in Theorem 5.1 is stably isomorphic to the determinant bundle gerbe constructed in Theorem 4.1, up to taking a product with a bundle gerbe with torsion Dixmier–Douady class. (The notion of products for bundle gerbes is covered in the original paper of [38].) This is because the Dixmier–Douady class of a product is the sum of the Dixmier–Douady classes and a bundle gerbe with torsion Dixmier–Douady class can be equipped with a connection and curving whose bundle gerbe curvature is trivial. Such a bundle gerbe is often called flat.

Remark 5.2. There is in fact an equivalent but different picture for the universal bundle gerbe. If a K^1-element in $K^1(B)$ is represented by a smooth family of self-adjoint Fredholm operators $\{T_b\}_{b\in B} \in \mathcal{F}_*^{a.s}$ on a Hilbert space H, there is a canonical principal $\mathbb{P}U(\mathcal{F}_H)$-bundle over B constructed as follows. First we work over $F_*^{a.s}$ by covering it with open sets of form

$$U_\lambda = \{T \in \mathcal{F}_*^{a.s} | \lambda \in \mathrm{spec}(T)\},$$

where $\lambda \in \mathbb{R}$. Over U_λ, there is a polarization

$$H = H_{T,\lambda}^- \oplus H_{T,\lambda}^+,$$

varying continuously in T given by the spectral decomposition of H into eigenspaces of $T \in U_\lambda$ corresponding to eigenvalues greater or less than λ. Denote by $P_{T,\lambda}$ the orthogonal projection onto $H_{T,\lambda}^-$. For $\lambda \neq \lambda'$, $P_{T,\lambda}$ and $P_{T,\lambda'}$ are related by conjugation by an element $g_{\lambda\lambda'}(T)$ of $\hat{\mathcal{U}}_{\mathrm{res}}$ from the lifting of a well-defined element

$$\tilde{g}_{\lambda\lambda'}(T) \in \mathcal{U}_{\mathrm{res}},$$

where the copy of $\hat{\mathcal{U}}_{\mathrm{res}}$ we are using is specified by a reference spectral projection of T. Now there is a Fock representation corresponding to this reference spectral projection and hence, using the inclusion $\mathcal{U}_{\mathrm{res}} \subset \mathbb{P}U(\mathcal{F}_H)$, a well-defined element

$$\tilde{g}_{\lambda\lambda'}(T) \in \mathbb{P}U(\mathcal{F}_H).$$

The family of functions $\{\tilde{g}_{\lambda\lambda'}\}$ defined on overlaps $U_{\lambda\lambda'}$ can be used as transition functions to construct a principal $\mathbb{P}U(\mathcal{F}_H)$-bundle over $F_*^{a.s}$. We get a corresponding bundle over B using the pull-back construction.

The above argument also gives a universal determinant bundle gerbe $\mathcal{D}$ over $F_*^{a.s}$ by using the construction in Section 4 via determinant line bundles $\mathrm{Det}_{\lambda\lambda'}$ over intersections $U_{\lambda\lambda'}$. Recall from [19] that for $\lambda < \lambda'$, the line over T is $\mathrm{Det}_{\lambda\lambda'}(T)$: the highest exterior power of the vector space spanned by eigenvectors of T corresponding to eigenvalues between λ and λ'. The Dixmier–Douady class of this determinant bundle gerbe can be determined exactly. To see this we observe that the Dixmier–Douady class of the pull-back of $\mathcal{D}$ to $\mathcal{U}_1$ using an explicit homotopy equivalence from [3] must be a multiple, say n, of the fundamental class on $\mathcal{U}_1$. To determine which multiple just take B to be S^3 and choose a family $\{T_b\}_{b\in S^3} \in \mathcal{F}_*^{a.s}$ to represent the generator of $H^3(\mathcal{F}_*^{a.s}, \mathbb{Z}) \cong \mathbb{Z}$ (an explicit example is given in the last section of [19]). Under the explicit homotopy equivalence from [3], we know that the resulting continuous map $S^3 \to \mathcal{U}_1$ also defines the generator of $H^3(\mathcal{U}_1, \mathbb{Z}) \cong \mathbb{Z}$. The fact that $n = 1$ follows from (15) as in the case of S^3 the curvature suffices to determine the Dixmier–Douady class.

6. Index gerbe as induced from the universal bundle gerbe

In this section, we will construct the index gerbe associated with a family of Dirac operators on an odd dimensional manifold with or without boundary. To do this we need to recall some additional standard material using the notation of Section 3.

Let $\pi : X \to B$ be a smooth fibration with even dimensional fibres. With a choice of spectral section P for the family of boundary Dirac operators $\{\not{D}_b^{\partial M}\}_{b\in B}$, we have a family of Fredholm operators $\{\not{D}^M_{+,P;b}\}_{b\in B}$ over B. There is a determinant line bundle, denoted by $\mathrm{Det}(\not{D}^M_{+,P})$, over B given by

$$\mathrm{Det}(\not{D}^M_{+,P})_b = \Lambda^{\mathrm{top}}(\mathrm{Ker}\,\not{D}^M_{+,P;b})^* \otimes \Lambda^{\mathrm{top}}(\mathrm{Coker}\,\not{D}^M_{+,P;b}).$$

Using zeta determinant regularization, as in [43,10,47,45], a Hermitian metric and a unitary connection $\nabla^{\not{D}^M_{+,P}}$ can be constructed on $\mathrm{Det}(\not{D}^M_{+,P})$ such that

$$c_1(\mathrm{Det}(\not{D}^M_{+,P}), \nabla^{\not{D}^M_{+,P}}) = [\pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)) - \tilde{\eta}^P_{\mathrm{even}}]_{(2)}, \tag{16}$$

Note that the eta form $\tilde{\eta}^P_{\mathrm{even}}$ is defined modulo exact forms, the above equality (16) holds modulo exact 2-forms.

Now we let $\pi : X \to B$ have closed even dimensional fibers partitioned into two codimension 0 submanifolds, $M = M_0 \cup M_1$, joined along a codimension 1 submanifold $\partial M_0 = -\partial M_1$. Assume that the metric $g^{X/B}$ is of product type near the collar neighborhood of the separating submanifold. Let P be a spectral section for $\{\not{D}_b^{\partial M_0}\}_{b\in B}$, then $I - P$ is a spectral section for $\{\not{D}_b^{\partial M_1}\}_{b\in B}$. Scott showed [44] that the determinant line bundles for these three families of $\{\not{D}^M_{+;b}\}_{b\in B}$, $\{\not{D}^{M_0}_{+,P;b}\}_{b\in B}$ and $\{\not{D}^{M_1}_{+,I-P;b}\}_{b\in B}$ satisfy the following gluing formulae as Hermitian line bundles:

$$\mathrm{Det}(\not{D}^M_+) \cong \mathrm{Det}(\not{D}^{M_0}_{+,P}) \otimes \mathrm{Det}(\not{D}^{M_1}_{+,I-P}), \tag{17}$$

moreover the splitting formula for the curvature of $\nabla^{\not{D}^M_+}$ implies that

$$c_1(\mathrm{Det}(\not{D}^M_+), \nabla^{\not{D}^M_+}) = c_1(\mathrm{Det}(\not{D}^{M_0}_{+,P}), \nabla^{\not{D}^{M_0}_{+,P}}) + c_1(\mathrm{Det}(\not{D}^{M_1}_{+,I-P}), \nabla^{\not{D}^{M_1}_{+,I-P}}). \tag{18}$$

With these facts in hand we move on to the main results of this paper.

6.1. Index gerbe from a family of Dirac operators on a closed manifold

Let $\pi : X \to B$ be a smooth fibration over a closed smooth manifold B, whose fibers are diffeomorphic to a compact, oriented, odd dimensional spin manifold M.

Let $g^{X/B}$ be a metric on the relative tangent bundle $T(X/B)$ and let $S_{X/B}$ be the spinor bundle associated with $(T(X/B), g^{X/B})$. Let T^HX be a horizontal vector subbundle of TX. Then $(T^HX, g^{X/B})$ determines a connection $\nabla^{X/B}$ on $T(X/B)$ as in Section 3. Let (V, h^V, ∇^V) be a Hermitian vector bundle over X equipped with a unitary connection.

The family of Dirac operators $\{\not{D}_b\}_{b\in B}$ defines a K^1-element

$$\mathrm{Ind}(\not{D}) \in K^1(B),$$

with $\mathrm{Ch}(\mathrm{Ind}(\not{D})) = \pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)) \in H^{\mathrm{odd}}(B, \mathbb{R})$ as given by the Atiyah–Singer index theorem (Theorem (7)).

Then Theorem 5.1 provides a canonical bundle gerbe $\mathcal{G}^M$ over B with bundle gerbe connection and curving whose curvature is given by

$$\pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V))_{(3)}.$$

Now we can prove the following theorem, which was obtained in [29] by Lott using a different method.

Theorem 6.1. *Let $\pi : X \to B$ be a smooth fibration with fibers diffeomorphic to closed odd dimensional spin manifolds M and V be a Hermitian vector bundle over X equipped with a unitary connection ∇^V. Then the associated family of Dirac operators defines a canonical bundle gerbe $\mathcal{G}^M$ over B equipped with a Hermitian metric and a unitary gerbe connection whose curving (up to an exact form) is given by the locally defined eta form such that its bundle gerbe curvature is given by*

$$\left(\int_M \hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)\right)_{(3)}.$$

Proof. Cover B by U_λ (for $\lambda \in \mathbb{R}$) such that

$$U_\lambda = \{b \in B \mid \not{D}_b - \lambda \text{ is invertible}\}.$$

Over U_λ, the bundle of Hilbert spaces of square integrable sections along the fibers has a continuous polarization:

$$H_b = H_b^+(\lambda) \oplus H_b^-(\lambda),$$

the spectral decomposition with respect to $\not{D}^M_{P,b} - \lambda$ into the positive and negative eigenspaces. Denote by P_λ the continuous family of orthogonal projections $\{P_{\lambda,b}\}_{b\in U_\lambda}$ onto $H_b^+(\lambda)$ along the above continuous polarization over U_λ.

Note that the restriction of $\mathrm{Ind}(\not{D})$ on U_λ is trivial by the result of Melrose and Piazza in [34], as $\not{D}$ over U_λ admits a spectral section. So over U_λ, its Chern character form is exact:

$$\pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)) = d\tilde{\eta}^{P_\lambda}_{\text{even}} \tag{19}$$

where $\tilde{\eta}^{P_\lambda}_{\text{even}}$, unique up to an exact form, is the even eta form on U_λ, associated with $\{\not{D}^M_b\}_{b\in U_\lambda}$ and the spectral section P_λ.

For $\lambda \geq \lambda'$, over $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$, consider the fibration $[0, 1] \times X \to B$ with even dimensional fibers, the boundary fibration has two components $\{0\} \times X$ and $\{1\} \times X$. The spectral section can be chosen to be

$$P_{\lambda\lambda'} = P_\lambda \oplus (I - P_{\lambda'}). \tag{20}$$

Associated with this spectral section $P_{\lambda\lambda'}$, there exists an even eta form $\tilde{\eta}^{P_{\lambda\lambda'}}_{\text{even}}$ on $U_{\lambda\lambda'}$, modulo exact forms, depending only on $P_{\lambda\lambda'}$ and the family of boundary Dirac operators. From the definition of the eta form, we have

$$\tilde{\eta}^{P_{\lambda\lambda'}}_{\text{even}} = \tilde{\eta}^{P_\lambda}_{\text{even}} - \tilde{\eta}^{P_{\lambda'}}_{\text{even}},$$

which should be understood modulo exact even forms on $U_{\alpha\beta}$. Then the family of Dirac operators (with boundary condition given by the spectral section $P_{\lambda\lambda'}$) denoted by

$$\{\not{D}^{[0,1]\times M}_{P_{\lambda\lambda'},b}\}_{b\in U_{\lambda\lambda'}},$$

is a smooth family of Fredholm operators, which has a well-defined index $\mathrm{Ind}(\not{D}^{[0,1]\times M}_{P_{\lambda\lambda'}})$ in $K^0(U_{\lambda\lambda'})$.

The corresponding determinant line bundle over $U_{\lambda\lambda'}$, denoted by $\mathrm{Det}_{\lambda\lambda'}$, is a Hermitian line bundle equipped with the Quillen metric and the Bismut–Freed unitary connection. Its first Chern class is given by the local family index formula (16):

$$[\pi_*(\hat{A}(T([0, 1] \times X/U_{\lambda\lambda'}), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)) - \tilde{\eta}^{P_{\lambda\lambda'}}_{\text{even}}]_{(2)}. \tag{21}$$

Note that in this situation, the contribution from the characteristic class

$$\pi_*(\hat{A}(T([0, 1] \times X/U_{\lambda\lambda'}), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V))$$

vanishes, as it has no component in the [0, 1]-direction. Hence, the first Chern class of $\mathrm{Det}_{\lambda\lambda'}$ is represented by the form

$$(\tilde{\eta}^{P_{\lambda\lambda'}}_{\mathrm{even}})_{(2)} = (\tilde{\eta}^{P_{\lambda'}}_{\mathrm{even}} - \tilde{\eta}^{P_{\lambda}}_{\mathrm{even}})_{(2)}.$$

Again these eta forms are well defined only modulo locally defined exact 2-forms. This then implies that the even eta form

$$(\tilde{\eta}^{P_{\lambda}}_{\mathrm{even}})_{(2)},$$

which is only defined over U_λ, is the curving (up to an exact 2-form) for the Bismut–Freed connection on $\mathrm{Det}_{\lambda\lambda'}$. Hence, the gerbe curvature is uniquely determined and given by

$$\mathrm{d}(\tilde{\eta}^{P_{\lambda}}_{\mathrm{even}})_{(2)} - \mathrm{d}(\tilde{\eta}^{P_{\lambda'}}_{\mathrm{even}})_{(2)} = \left(\int_M \hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)\right)_{(3)}. \quad \square$$

6.2. Index gerbe from a family of Dirac operators on a manifold with boundary

Now we assume that the fibration $\pi : X \to B$ has fiber diffeomorphic to an odd dimensional *Spin* manifold M with non-empty boundary.

A $Cl(1)$-spectral section P for the family of boundary Dirac operators $\{\not{D}^{\partial M}_b\}_{b\in B}$ provides a well-defined index for the family of self-adjoint Fredholm operators $\{\not{D}^M_P\}$:

$$\mathrm{Ind}(\not{D}^M_P) \in K^1(B),$$

with $\mathrm{Ch}(\mathrm{Ind}(\not{D})) = \pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V) - \tilde{\eta}^P_{\mathrm{odd}}) \in H^{\mathrm{odd}}(B, \mathbb{R})$ as given by the local family index theorem in Theorem 3.3.

Then Theorem 5.1 provides a canonical bundle gerbe $\mathcal{G}^M_P$ over B with a bundle gerbe connection and curving whose curvature is given by:

$$(\pi_*(\hat{A}(T(X/B), \nabla^{X/B})\mathrm{Ch}(V, \nabla^V)) - \tilde{\eta}^P_{\mathrm{odd}})_{(3)}. \tag{22}$$

Here we should emphasize that $\tilde{\eta}^P_{\mathrm{odd}}$ is an odd eta form associated with a perturbation of $\{\not{D}^{\partial M}_b\}_{b\in B}$ by a family of self-adjoint smoothing operators $\{A_{P,b}\}_{b\in B}$ such that $\not{D}^{\partial M}_b + A_{P,b}$ is invertible as in [35]. Note that, as previously, we always define $\tilde{\eta}^P_{\mathrm{odd}}$ up to an exact form.

We aim to understand the bundle gerbe connection and its curving such that the bundle gerbe curvature is given by (22). Note that the argument for the previous case cannot be applied here, as now $[0, 1] \times M$ is a manifold with corners near ∂M, and to our knowledge the even eta form for a family of Dirac operators on odd dimensional manifolds with boundary which transgresses the odd local index form (22) has not been found. For a manifold with corners, Fredholm perturbations of Dirac operators and their index formulae have been developed by Loya and Melrose [31] but for a family of Dirac operators on a manifold with corners up to codimension 2 their theory does not apply.

Instead, we will apply the theory of bundle gerbes to find this even eta form which transgresses the odd local index form (22) and discuss its implications for local family index theory for a family of Fredholm operators on a manifold with corners of a particular type, $[0, 1] \times M$.

For the fibration $\pi : X \to B$, whose fibers are odd dimensional manifolds with non-empty boundary, the Melrose and Piazza $Cl(1)$-spectral section P for the family of boundary Dirac operators $\{\not{D}^{\partial M}_b\}_{b\in B}$ gives rise to a family of self-adjoint Fredholm operators $\{\not{D}^M_{P,b}\}_{b\in B}$ with discrete spectrum.

Cover B by U_λ with $\lambda \in \mathbb{R}$ such that

$$U_\lambda = \{b \in B | \not{D}^M_{P,b} - \lambda \text{ is invertible}\}.$$

Lemma 6.2. *Over U_λ, the family of self-adjoint Fredholm operators $\{\not{D}^M_P\}$ has trivial index in $K^1(U_\lambda)$.*

Proof. This follows from the fact that, over U_λ, we have a uniform polarization of the Hilbert space of square integrable sections along the fibers with boundary condition given by the $Cl(1)$-spectral section P,

$$H_{P,b} = H^+_{P,b}(\lambda) \oplus H^-_{P,b}(\lambda),$$

the spectral decomposition with respect to $\not{D}^M_{P,b} - \lambda$ into the positive and negative spectral subspaces. Denote by P_λ the smooth family of orthogonal projections $\{P_{\lambda,b}\}_{b\in U_\lambda}$ onto $H^+_{P,b}(\lambda)$ along the above uniform polarization over U_λ. Over U_λ, $\not{D}^M_P$ admits a spectral section $\{P_{\lambda,b}\}_{b\in U_\lambda}$, by the result of Melrose and Piazza in [34], $\{\not{D}^M_P\}$ has trivial index in $K^1(U_\lambda)$. □

Hence, there exists an even form $\eta^{P,\lambda}_{\text{even}}$ on U_λ, unique up to an exact form, satisfying the following transgression formula over U_λ:

$$d\eta^{P,\lambda}_{\text{even}} = \pi_*(\hat{A}(T(X/B), \nabla^{X/B})\text{Ch}(V, \nabla^V)) - \tilde{\eta}^P_{\text{odd}}. \tag{23}$$

We remark that these forms $\{\eta^{P,\lambda}_{\text{even}}\}$ should in fact be even eta forms *a la* Melrose and Piazza's construction in [35] associated with the family of self-adjoint Fredholm operators $\{\not{D}^M_{P,b}\}_{b\in U_\lambda}$ and the spectral section P_λ over U_λ. We do not have a proof however.

From the abstract bundle gerbe theory in [38], we know that this locally defined even form $(\eta^{P,\lambda}_{\text{even}})_{(2)}$ on U_λ is the curving, up to an exact form on U_λ, for the bundle gerbe induced from the universal gerbe as in Theorem 5.1. Hence, we have established the following Theorem.

Theorem 6.3. *Let $\pi : X \to B$ be a fibration with fiber diffeomorphic to an odd dimensional Spin manifold M with non-empty boundary. Let P be a $Cl(1)$-spectral section such that $\{\not{D}^M_P\}$ is a continuous family of self-adjoint Fredholm operators parametrized by B, then the associated bundle gerbe $\mathcal{G}^M_P$ over B can be equipped with a bundle gerbe connection and curving $\{(\eta^{P,\lambda}_{\text{even}})_{(2)}\}$ with the curvature given by*

$$d(\eta^{P,\lambda}_{\text{even}})_{(2)} = \left(\int_M \hat{A}(T(X/B), \nabla^{X/B})\text{Ch}(V, \nabla^V) - \tilde{\eta}^P_{\text{odd}}\right)_{(3)}.$$

6.3. Index gerbe from a pair of $Cl(1)$-spectral sections

From Proposition 4 in [35], we know that if P_1 and P_2 are two $Cl(1)$-spectral sections for $\{\not{D}^{\partial M}_b\}_{b\in B}$, then the Atiyah–Singer suspension operation [3] defines a difference element

$$[P_2 - P_1] \in K^1(B),$$

such that

$$\text{Ind}(\not{D}^M_{P_1}) - \text{Ind}(\not{D}^M_{P_2}) = [P_2 - P_1] \in K^1(B),$$

whose Chern character is given by $\tilde{\eta}^{P_2}_{\text{odd}} - \tilde{\eta}^{P_1}_{\text{odd}}$. It was also shown in Proposition 12 in [35] that for a fixed P_1, as P_2 ranges over all $Cl(1)$-spectral sections, $[P_2 - P_1]$ exhausts $K^1(B)$.

Apply Theorem 5.1 again to $[P_2 - P_1]$, we obtain a canonical bundle gerbe $\mathcal{G}_{P_2P_1}$ associated with $[P_2 - P_1] \in K^1(B)$, with a bundle gerbe connection and curving whose curvature is given by

$$(\tilde{\eta}^{P_2}_{\text{odd}} - \tilde{\eta}^{P_1}_{\text{odd}})_{(3)}.$$

Moreover, we have the following stable isomorphism relating the bundle gerbes associated with families of Dirac operators on odd dimensional manifolds with boundary equipped with two different $Cl(1)$-spectral sections for the family of boundary Dirac operators

$$\mathcal{G}^M_{P_1} \cong \mathcal{G}^M_{P_2} \otimes \mathcal{G}_{P_2P_1}.$$

Take a smooth fibration X over B, with closed odd dimensional fibers partitioned into two codimensional 0 submanifolds $M = M_0 \cup M_1$ along a codimension 1 submanifold $\partial M_0 = -\partial M_1$. Assume that the metric $g^{X/B}$ is of product type near the collar neighborhood of the separating submanifold. Let P be a $Cl(1)$-spectral section for $\{\not{D}^{\partial M_0}_b\}_{b\in B}$, then $I - P$ is a $Cl(1)$-spectral section for $\{\not{D}^{\partial M_1}_b\}_{b\in B}$. Then we have the following splitting formula for the canonical bundle gerbe $\mathcal{G}^M$ obtained in Theorem 6.1:

$$\mathcal{G}^M \cong \mathcal{G}^{M_0}_P \otimes \mathcal{G}^{M_1}_{I-P}.$$

Acknowledgements

ALC acknowledges advice from P. Piazza on spectral sections and also thanks V. Mathai for conversations on his joint work with R. Melrose related to the theme analysed in this paper. BLW wishes to thank Xiaonan Ma for many invaluable comments and his explanations of eta forms.

This work was supported by the Australian Research Council Discovery Project DP0449470.

References

[1] M. Atiyah, K-theory, W.A. Benjamin, New York, 1967.
[2] M.F. Atiyah, V.K. Patodi, I.M. Singer, Spectral asymmetry and Riemannian geometry, I, II, III, Math. Proc. Cambridge Philos. Soc. 77 (1975) 43–69; 78 (1975) 405–432; 79 (1976) 71–99.
[3] M.F. Atiyah, I.M. Singer, Index theory for skew-adjoint Fredholm operators, Inst. Hautes tudes Sci. Publ. Math. (37) (1969) 5–26.
[4] M.F. Atiyah, I.M. Singer, Dirac operators coupled to vector potentials, Natl. Acad. Sci. (USA) 81 (1984) 2597.
[5] N. Berline, E. Getzler, M. Vergne, Heat kernels and the Dirac operators, 1992.
[6] J.M. Bismut, Local index theory, eta invariants and holomorphic torsion: A survey, in: Surveys in Differential Geometry, Vol. III (Cambridge, MA, 1996), Int. Press, Boston, MA, 1998, pp. 1–76.
[7] J.-M. Bismut, The index Theorem for families of Dirac operators: Two heat equation proofs, Invent. Math. 83 (1986) 91–151.
[8] J.M. Bismut, J. Cheeger, η-invariants and their adiabatic limits, J. Amer. Math. Soc. 2 (1) (1989) 33–70.
[9] J.M. Bismut, J. Cheeger, Families index for manifolds with boundary, superconnections and cones I and II, J. Funct. Anal. 89 (1990) 313–363; 90 (1990) 306–354.
[10] J.-M. Bismut, D. Freed, The analysis of elliptic families. II. Eta invariants and the holonomy theorem, Comm. Math. Phys. 107 (1986) 103–163.
[11] B. Boo-Bavnbek, K. Wojciechowski, Elliptic Boundary Problems for Dirac Operators, Birkhauser, Boston, 1993.
[12] P. Bouwknegt, A. Carey, V. Mathai, M. Murray, D. Stevenson, Twisted K-theory and K-theory of bundle gerbes, Comm. Math. Phys. 228 (1) (2002) 17–45.
[13] P. Bouwknegt, V. Mathai, D-branes, B-fields and twisted K-theory, J. High Energy Phys. 03 (2000) 007. hep-th/0002023.
[14] J.-L. Brylinski, Loop spaces, characteristic classes and geometric quantization, in: Progress in Mathematics, vol. 107, Birkhauser, Boston, 1993.
[15] U. Bunke, Transgression of the index gerbe, Manuscripta Math. 109 (3) (2002) 263–287.
[16] U. Bunke, Index theory, eta forms, and Deligne cohomology (preprint).
[17] U. Bunke, J. Park, Determinant bundles, boundaries and surgery, J. Geom. Phys. 52 (2004) 28–43.
[18] A.L. Carey, M.K. Murray, Faddeev's anomaly and bundle gerbes, Lett. Math. Phys. 37 (1996).
[19] A. Carey, J. Mickelsson, M. Murray, Index theory, gerbes, and Hamiltonian quantization, Comm. Math. Phys. 183 (1997) 707–722.
[20] A. Carey, J. Mickelsson, M. Murray, Bundle gerbes applied to quantum field theory, Rev. Math. Phys. 12 (2000) 65–90.
[21] A. Carey, J. Mickelsson, A gerbe obstruction to quantization of fermions on odd dimensional manifolds, Lett. Math. Phys. 51 (2000) 145–160.
[22] A.L. Carey, J. Mickelsson, The universal gerbe, Dixmier–Douady classes and gauge theory, Lett. Math. Phys. 59 (2002) 47–60.
[23] A.L. Carey, M.K. Murray, B.L. Wang, Higher bundle gerbes and cohomology classes in gauge theories, J. Geom. Phys. 21 (1997) 183–197.
[24] C. Ekstrand, J. Mickelsson, Gravitational anomalies, gerbes, and hamiltonian quantization.
[25] K. Gawedski, N. Reis, WZW branes and gerbes, Rev. Math. Phys. 14 (12) (2002) 1281–1334.
[26] E. Getzler, The odd Chern character in cyclic homology and spectral flow, Topology 32 (1993) 489–507.
[27] N. Hitchin, Lectures on special Lagrangian submanifolds, in: Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), in: AMS/IP Stud. Adv. Math., 23, Amer. Math. Soc., Providence, RI, 2001, pp. 151–182.
[28] M.J. Hopkins, I.M. Singer, Quadratic functions in geometry, topology, and M-theory, preprint: math.AT/0211216.
[29] J. Lott, Higher-degree analogs of the determinant line bundle, Comm. Math. Phys. 230 (1) (2002) 41–69.
[30] P. Loya, The Index of b-pseudodifferential Operators on Manifolds with Corners (preprint).
[31] P. Loya, R. Melrose, Fredholm Perturbations of Dirac Operators on Manifolds with Corners (preprint).
[32] Private communication.
[33] R. Mazzeo, P. Piazza, Dirac operators, heat kernels and microlocal analysis. II. Analytic surgery, Rend. Mat. Appl. (7) 18 (2) (1998) 221–288.
[34] R.B. Melrose, P. Piazza, Families of Dirac operators, boundaries and the b-calculus, J. Differential Geom. 146 (1997) 99–180.
[35] R.B. Melrose, P. Piazza, An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary, J. Differential Geom. 146 (1997) 287–334.
[36] J. Mickelsson, Gerbes, twisted K-theory, and the supersymmetric WZW model, hep-th/0206139.
[37] J. Mickelsson, S.G. Rajeev, Current algebras in d+1 dimensions and determinant line bundles over infinite-dimensional Grassmannians, Comm. Math. Phys. 116 (1988) 400.
[38] M.K. Murray, Bundle gerbes, J. London Math. Soc. (2) 54 (2) (1996) 403–416.
[39] M.K. Murray, D. Stevenson, Higgs fields, bundle gerbes and string structures, Comm. Math. Phys. 243 (3) (2003) 541–555.
[40] P. Piazza, Determinant bundles, manifolds with boundary and surgery, Comm. Math. Phys. 178 (3) (1996) 597–626.
[41] P. Piazza, Determinant bundles, manifolds with boundary and surgery. II. Spectral sections and surgery rules for anomalies, Comm. Math. Phys. 193 (1) (1998) 105–124.
[42] A. Pressley, G. Segal, Loop Groups, Clarendon Press, Oxford, 1986.

[43] D. Quillen, Determinants of Cauchy–Riemann operators on Riemann surfaces, Funct. Anal. Appl. 14 (1985) 31–34.
[44] S. Scott, Splitting the curvature of the determinant line bundle, Proc. Amer. Math. Soc. 128 (9) (2000) 2763–2775.
[45] S. Scott, K. Wojciechowski, The ζ-determinant and Quillen determinant for a Dirac operator on a manifold with boundary, Geom. Funct. Anal. 10 (5) (2000) 1202–1236.
[46] G.B. Segal, Faddeev's anomaly in Gauss' law, 1986, unpublished manuscript.
[47] K. Wojciechowski, The ζ-determinant and the additivity of the η-invariant on the smooth, self-adjoint Grassmannian, Comm. Math. Phys. 201 423–444.